Rappel : Techniques d'intégration

Proposition. Formule de changement de variable.

$f : [a,b] \to \mathbb{R}$ continue, $\varphi : [\alpha,\beta] \to [a,b]$ continûment dérivable sur $I \supset [\alpha,\beta]$.

Alors
$$\boxed{\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f(\varphi(t))\cdot\varphi'(t)\,dt}$$
où $x = \varphi(t)$

Dém : Soit $G(t) = \int_a^{\varphi(t)} f(x)\,dx \Rightarrow G'(t) = f(\varphi(t))\cdot\varphi'(t) := g(t)$

$\Rightarrow G(t)$ est une primitive de $g(t)$ sur $[\alpha,\beta]$ $\Rightarrow \int_\alpha^\beta g(t)\,dt = G(\beta) - G(\alpha)$

$\varphi : [\alpha,\beta] \to [a,b]$

$$\int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt = \int_a^{\varphi(\beta)} f(x)\,dx - \int_a^{\varphi(\alpha)} f(x)\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

$$\int_a^{\varphi(\beta)} f(x)\,dx + \int_{\varphi(\alpha)}^{a} f(x)\,dx$$

$\blacksquare$

Ex2. (1) $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}} \underset{\substack{x=\sin t=\varphi(t)\\ dx=\varphi'(t)dt=\cos t\,dt\\ t=\operatorname{Arcsin} x}}{=} \int \frac{\sin t \cos t}{\cos t}\,dt = \int \sin t\,dt = -\cos t + C = -\cos(\operatorname{Arcsin} x) + C =$

$= -\sqrt{1-\sin^2(\operatorname{Arcsin} x)} + C = -\sqrt{1-x^2} + C$

intégrale indéfinie = la plus générale primitive

(2) $\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = -\frac{1}{2}\,\frac{1}{\frac{1}{2}}\,u^{\frac{1}{2}} + C = -u^{\frac{1}{2}} + C = -\sqrt{1-x^2} + C$ ☺

$x\,dx = -\frac{1}{2}du$

$u = 1-x^2 \Rightarrow du = -2x\,dx$

Exercice: $\displaystyle\int_0^{1/2} \frac{x\,dx}{\sqrt{1-x^2}} \underset{t\in[0,\frac{\pi}{6}]}{\overset{x=\sin t}{=}} \qquad = -\frac{\sqrt{3}}{2} + 1.$

Ex3. $\displaystyle\int \frac{dx}{\sin x} \longrightarrow$ (1) $\displaystyle\int \frac{\sin x}{\sin^2 x}\,dx = -\int \frac{d\cos x}{1-\cos^2 x} \overset{u=\cos x}{=} -\int \frac{du}{1-u^2} = -\frac{1}{2}\int\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du =$

$-u$

$\displaystyle = \frac{1}{2}\log|1-u| - \frac{1}{2}\log|1+u| + C = \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| + C$

$\begin{cases} 1-\cos x = 2\sin^2\frac{x}{2} \\ 1+\cos x = 2\cos^2\frac{x}{2} \end{cases}$

(2) $\displaystyle\int \frac{\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \frac{1}{2}\int \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\,dx + \frac{1}{2}\int \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\,dx =$

$\displaystyle = \int \frac{\sin\frac{x}{2}\,d\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} + \int \frac{\cos\frac{x}{2}\,d\left(\frac{x}{2}\right)}{\sin\frac{x}{2}} = -\int \frac{d\left(\cos\frac{x}{2}\right)}{\cos\frac{x}{2}} + \int \frac{d\left(\sin\frac{x}{2}\right)}{\sin\frac{x}{2}} = -\log\left|\cos\frac{x}{2}\right| + \log\left|\sin\frac{x}{2}\right| + C$

Remarque. Parfois deux expressions différentes sont égales à une constante près.

$$\log 5x = \log x + C \qquad : \qquad \log 5x = \log x + \log 5$$

$$\frac{1}{\cos^2 t} = \operatorname{tg}^2 t + C \qquad : \qquad \operatorname{tg}^2 t = \frac{\sin^2 t}{\cos^2 t} = \frac{1-\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} - 1$$

Calcul des dérivée: =

Calcul des intégrales =

Proposition. Formule d'intégration par parties.

$g, f : I \to \mathbb{R}$ continûment dérivable ; $[a,b] \subset I$. Alors

$$\boxed{\int_a^b f(x)\, g'(x)\, dx = f(x)\, g(x)\Big|_a^b - \int_a^b g(x)\, f'(x)\, dx.}$$

Dém: $(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\, g'(x) \Rightarrow f(x)\cdot g'(x) = (f\cdot g)'(x) - f'(x)\cdot g(x)$

$$\Rightarrow \int_a^b f(x)\cdot g'(x)\, dx = \int_a^b (f\cdot g)'(x)\, dx - \int_a^b f'(x)\cdot g(x)\, dx = f(x)\cdot g(x)\Big|_a^b - \int_a^b g(x)\cdot f'(x)\, dx.$$

primitive de $(fg)'$

$\blacksquare$

$$\boxed{\int_a^b f\, dg = fg\Big|_a^b - \int_a^b g\, df}$$

$$\boxed{df(x) = f'(x)\, dx}$$

Ex 4. $\int \underbrace{(\log x)^2}_{f(x)} \, d\underbrace{x}_{g(x)} = x(\log x)^2 - \int \underbrace{\left(2\log x\right)\cdot\frac{1}{x}}_{f'(x)} \cdot \underbrace{x}_{g(x)} \, dx = x(\log x)^2 - 2\int \underbrace{\log x}_{f(x)} \, d\underbrace{x}_{g(x)} =$

$= x(\log x)^2 - 2x\log x + 2\int x\cdot\frac{1}{x}\,dx = x(\log x)^2 - 2x(\log x) + 2x + C$

Marche bien pour (polynômes)$\cdot(\log x)^k$, (polynômes)$(\sin x, \cos x)$, (polynômes)$\cdot e^x$

Ex 5. $\int \sin x \, e^x \, dx = \int \underbrace{\sin x}_{f(x)} \, d\underbrace{e^x}_{g(x)} = e^x \sin x - \int \underbrace{e^x}_{g(x)} \underbrace{\cos x}_{f'(x)} \, dx = e^x \sin x - \int e^x \, d(\sin x) =$

$(e^x)' dx = d(e^x)$

$= e^x \cancel{\sin x} - e^x \cancel{\sin x} + \int \sin x \cdot e^x \, dx = \int \sin x \, e^x \, dx$ ☹

$e^x \sin x - \int \underbrace{\cos x}_{f(x)} \, d\underbrace{e^x}_{g(x)} = e^x \sin x - e^x \cos x + \int \underbrace{e^x}_{g(x)} \underbrace{(-\sin x)}_{f'(x)} \, dx = e^x(\sin x - \cos x) - \int e^x \sin x \, dx$

$\Rightarrow \int \sin x \, e^x \, dx = e^x(\sin x - \cos x) - \int e^x \sin x \, dx$

$\Rightarrow \int \sin x \, e^x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$ ☺

Marche pour $e^x(\sin x, \cos x)$

Remarque. Changement de variable: $f'(x)dx = d(f(x)) = du$, si $u = f(x)$.

Ex: $\int poly \cdot \cos x \, dx = \int poly \, d(\sin x)$; $\int g(x) f'(x) dx = \int g(x) d(f(x))$

Ex 6. $\int \frac{\log\sqrt{x}}{(x-1)^2} dx = \frac{1}{2}\int \frac{\log x}{(x-1)^2} dx = -\frac{1}{2}\int \log x \cdot \left(\frac{1}{x-1}\right)' dx = -\frac{1}{2}\int \underbrace{\log x}_{f(x)} \, d\Big(\underbrace{\frac{1}{x-1}}_{g(x)}\Big) =$

$= -\frac{1}{2}\frac{\log x}{x-1} + \frac{1}{2}\int \frac{1}{x-1}\cdot\frac{1}{x} dx = -\frac{1}{2}\frac{\log x}{x-1} + \frac{1}{2}\int\left(\frac{1}{x-1} - \frac{1}{x}\right) dx =$

$= \underbrace{-\frac{1}{2}\frac{\log x}{x-1} + \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x| + C}$

Exercice: Vérifier par dérivation: $\Big(\quad \Big)' = \frac{1}{2}\frac{\log x}{(x-1)^2}$.

Intégration des fonctions rationnelles. $\int \frac{P(x)}{Q(x)} dx$ s'exprime toujours en termes des fonctions élémentaires.

(Par contre, $\int \sin x \cdot \log x \, dx$ ne s'exprime pas en termes des fonctions élémentaires).

(1) $\int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{d(ax+b)}{ax+b} = \frac{1}{a}\log|ax+b| + C$

$a \neq 0$

(2) $\displaystyle\int \frac{(cx+d)\,dx}{(x-a)(x-b)} = \int \left(\frac{A}{x-a} + \frac{B}{x-b}\right)dx = A\log|x-a| + B\log|x-b| + C$

$a \neq b$

Coefficients indéterminés : $A(x-b)+B(x-a) = cx+d$

$$\Rightarrow \left.\begin{array}{ll} x^0: & -Ab-Ba = d \\ x^1: & A+B = c \end{array}\right\} \Rightarrow A = \frac{ac-d}{a-b}\ ;\ B = c-A = \dots$$

(3) $\displaystyle\int \frac{dx}{(ax+b)^k} \underset{\substack{t=ax+b \\ dt=a\,dx}}{=} \frac{1}{a}\int \frac{dt}{t^k} = \frac{1}{a}\int t^{-k}\,dt = \frac{1}{a}\frac{1}{-k+1}t^{-k+1} + C = \frac{1}{a(1-k)}(ax+b)^{-k+1} + C.$

$k \geq 2$ $\qquad$ $k \geq 2$

(4) $\displaystyle\int \frac{dx}{x^2+px+q} = \int \frac{dx}{\underbrace{\left(x+\frac{p}{2}\right)^2}_{u} + \underbrace{q-\frac{p^2}{4}}_{=\,c^2>0}} = \int \frac{du}{u^2+c^2} = \frac{1}{c^2}\int \frac{c\,d\left(\frac{u}{c}\right)}{\left(\frac{u}{c}\right)^2+1} \underset{t=\frac{u}{c}}{=} \frac{1}{c}\int \frac{dt}{t^2+1} =$

$p^2-4q<0$

$u = x+\frac{p}{2} \Rightarrow du = dx$

$$= \frac{1}{c}\,\mathrm{Arctg}\, t + C = \frac{1}{\sqrt{q-\frac{p^2}{4}}}\,\mathrm{Arctg}\left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}\right) + C.$$

(5) $\displaystyle\int \frac{x\,dx}{x^2+px+q} = \int \frac{\left(x+\frac{p}{2}\right)-\frac{p}{2}}{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}\,dx \underset{u=x+\frac{p}{2}}{=} \int \frac{\left(u-\frac{p}{2}\right)du}{u^2+c^2} = \int \frac{u\,du}{u^2+c^2} - \frac{p}{2}\int \frac{du}{u^2+c^2}$

Voir (4)

$p^2-4q<0$

$$\int \frac{u\,du}{u^2+c^2} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| + C = \frac{1}{2}\log|u^2+c^2| + C = \frac{1}{2}\log|x^2+px+q| + C.$$

$t = u^2+c^2 \Rightarrow dt = 2u\,du$

(6) $$\int \frac{dx}{(1+x^2)^n} = \int \frac{1}{\left(\frac{1}{\cos^2 t}\right)^n}\cdot\frac{1}{\cos^2 t}\,dt = \int (\cos t)^{2(n-1)}\,dt = \ldots$$

$x = \operatorname{tg} t \Rightarrow 1+\operatorname{tg}^2 t = \frac{1}{\cos^2 t}$

$dx = \frac{1}{\cos^2 t}\,dt$

$n \geq 2$

$(\cos t)^n = \left(\frac{e^{it}+e^{-it}}{2}\right)^n =$ cos, sin des angles multiples

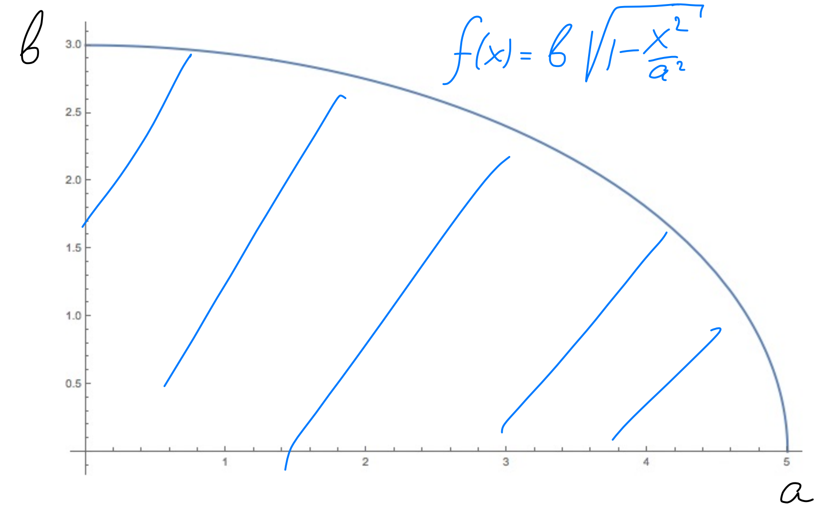
s'exprime en termes des sin, cos d'angle multiple

par exemple: $\cos^2 t = \frac{1}{2}(1+\cos 2t)$

(7) $$\int \frac{x\,dx}{(1+x^2)^n} = \frac{1}{2}\int\frac{du}{u^n} = \frac{1}{2}\,\frac{1}{-n+1}\,u^{-n+1} + C = \frac{1}{2(1-n)}(x^2+1)^{-n+1} + C.$$

$u = 1+x^2 \Rightarrow du = 2x\,dx$

Exemple. Aire d'une ellipse.



$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \Rightarrow y^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$\Rightarrow y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$\frac{1}{4}\text{Aire}(\text{ellipse}) = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = ab\int_0^1 \sqrt{1-u^2}\,du =$$

$u = \frac{x}{a}$; $du = \frac{1}{a}dx$

$$= a\cdot b\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,\cos t\,dt = ab\int_0^{\pi/2}\cos^2 t\,dt = a\cdot b\,\frac{1}{2}\int_0^{\pi/2}(1+\cos 2t)\,dt = \frac{1}{2}ab\,t\Big|_0^{\frac{\pi}{2}} + \underbrace{\frac{ab}{2}\cdot\frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{2}}}_{=0} =$$

$u = \sin t$

$du = \cos t\,dt$, $u \in [0,1] \Rightarrow t \in [0, \frac{\pi}{2}]$

$$= \frac{ab}{2}\cdot\frac{\pi}{2} - 0 = \frac{1}{4}\pi ab.$$

$$\Rightarrow \text{Aire}(\text{ellipse}) = \pi ab.$$